

# Moment-Based Spectral Analysis of Random Graphs with Given Expected Degrees

Victor M. Preciado, *Member, IEEE*, and M. Amin Rahimian, *Student, IEEE*

**Abstract**—In this paper, we analyze the limiting spectral distribution of the adjacency matrix of a random graph ensemble, proposed by Chung and Lu, in which a given expected degree sequence  $\bar{w}_n^T = (w_1^{(n)}, \dots, w_n^{(n)})$  is prescribed on the ensemble. Let  $\mathbf{a}_{i,j} = 1$  if there is an edge between the nodes  $\{i, j\}$  and zero otherwise, and consider the normalized random adjacency matrix of the graph ensemble:  $\mathbf{A}_n = [\mathbf{a}_{i,j}/\sqrt{n}]_{i,j=1}^n$ . The empirical spectral distribution of  $\mathbf{A}_n$  denoted by  $\mathbf{F}_n(\cdot)$  is the empirical measure putting a mass  $1/n$  at each of the  $n$  real eigenvalues of the symmetric matrix  $\mathbf{A}_n$ . Under some technical conditions on the expected degree sequence, we show that with probability one,  $\mathbf{F}_n(\cdot)$  converges weakly to a deterministic distribution  $F(\cdot)$ . Furthermore, we fully characterize this distribution by providing explicit expressions for the moments of  $F(\cdot)$ . We apply our results to well known degree distributions, such as power-law and exponential. The asymptotic expressions of the spectral moments in each case provide significant insights about the bulk behavior of the eigenvalue spectrum.

**Index Terms**—Complex Networks, Random Graph Models, Spectral Graph Theory, Random Matrix Theory.

## 1 INTRODUCTION

UNDERSTANDING the relationship between structural and spectral properties of a network is a key question in the field of Network Science. Spectral graph methods (see [1]–[3], and references therein) have become a fundamental tool in the analysis of large complex networks, and related disciplines, with a broad range of applications in machine learning, data mining, web search and ranking, scientific computing, and computer vision. Studying the relationship between the structure of a graph and its eigenvalues is the central topic in the field of *algebraic graph theory* [1]–[5]. In particular, the eigenvalues of matrices representing the graph structure, such as the adjacency or the Laplacian matrices, have a direct connection to the behavior of several networked dynamical processes, such as spreading processes [6], synchronization of oscillators [7], random walks [8], consensus dynamics [9], and a wide variety of distributed algorithms [10].

The availability of massive databases describing a great variety of real-world networks allows researchers to explore their structural properties with great detail. Statistical analysis of empirical data has unveiled the existence of multiple common patterns in a large variety of network properties, such as power-law degree distributions [11], or the small-world phenomenon [12]. Random graphs models are the tool-of-choice to analyze the connection between structural and spectral network properties. Aiming to replicate empirical observations, a variety of synthetic network models has been proposed in the literature [11], [12]. The structural property that have (arguably) attracted the most attention is the degree distribution. Empirical studies show that the degree distribution of important real-world networks, such as the Internet [13], Facebook, or Twitter, are heavy-tailed and can be approximated using power-law distributions

[14], [15].

Random graph models, such as the Erdős-Rényi [16], [17], the scale-free [15], and the small-world models [18], are a versatile tool for investigating the properties of real-world networks. We find in the literature several random graphs able to model degree distributions from empirical data. One of best known graphs is the configuration model, originally proposed by Bender and Canfield in [19]. This model is able to fit a given degree sequence *exactly* (under certain technical conditions). Although many structural properties of this model have been studied in depth [20], [21], it is not specially amenable to spectral analysis. In contrast, the preferential attachment model proposed in [15] provides a justification for the emergence of power-law degree distributions in real-world networks. A tractable alternative to the preferential attachment model was proposed by Chung and Lu in [22], and analyzed in [23]–[25]. In this model, which we refer to as the Chung-Lu model, an *expected* degree sequence is prescribed onto a random graph ensemble, that can be algebraically described using a (random) adjacency matrix.

Studies of the statistical properties of the eigenvalues of random graphs and networks are prevalent in many applied areas. Examples include the investigations of the spacing between nearest eigenvalues in random models [26], [27], as well as real-world networks [28]–[30]. Empirical observations highlight spectral features not observed in classical random matrix ensembles, such as a triangle-like eigenvalue distribution in power-law networks [31], [32] or an exponential decay in the tails of the eigenvalue distribution [33], [34].

### 1.1 Main Contributions

In this work we offer an exact characterization for the eigenvalue spectrum of the adjacency matrix of random graph models with a given expected degree sequence. This characterization is in terms of the moments of the eigenvalue

• The authors are with the Department of Electrical and Systems Engineering at the University of Pennsylvania, Philadelphia, PA 19104.

distributions and it hinges upon application of the moments method for identifying the limiting spectral distribution of a related sequence of random matrices that is constructed from centralizing and normalizing the graph adjacencies. Accordingly, we give closed form expressions that describe the almost sure limits of the spectral moments and can be used to draw important conclusions about the graph spectrum and to bound several quantities of interest related to the eigenvalue distribution of the graph adjacencies.

The remainder of this paper is organized as follows. Preliminaries on the background and motivation of our study, as well as the random graph model under consideration, are presented in Section 2. Our main results on the asymptotic spectral moments of the adjacencies of random graphs and the characterization of the limiting spectral distributions for the normalized adjacencies are presented in Section 3, where we also include an outline of the proofs (which are presented in detail in Section 7). In Section 4, we apply our results to a case where node degrees are obtained by random samples from the support of a preset function and show how our main results can be applied in analysis of the spectrum of large random graphs. We consider random graphs with an exponential degree distribution as a special case and derive the asymptotic expressions of its spectral moments. In Section 5 we consider the important case of power-law degree distribution, which is known to be a good descriptor for many real-world networks. The asymptotic expressions of the spectral moments of power-law networks allow us to analyze the bulk behavior of the eigenvalue spectrum in much greater details; in particular, we can quantify and characterize the similarities with and deviations from the triangular distribution that is reported in the literature [31], [32]. Section 6 concludes the paper.

## 2 BACKGROUND & MOTIVATION

### 2.1 Chung-Lu Random Graph Model

We consider the Chung-Lu random graph model introduced in [22] and analyzed in [23]–[25], in which an expected degree sequence given by the  $n$  non-negative entries of the vector  $\bar{w}_n^T = (w_1^{(n)}, \dots, w_n^{(n)})$  is prescribed over the set of nodes, labeled by  $[n]$ , of the graph ensemble.<sup>1</sup> In this model, each random edge is realized independently of all other edges and in accordance with the probability measure  $\mathbb{P}\{\cdot\}$ , specified below. Let  $\mathbb{E}\{\cdot\}$  and  $\text{Var}\{\cdot\}$  be the expectation and variance operators corresponding to  $\mathbb{P}\{\cdot\}$ . Following [35] we allow for self-loops. To each random graph, we associate a (random) adjacency matrix, which is a zero-one matrix with the  $(i, j)$ -th entry being one if, and only if, there is an edge connecting nodes  $i$  and  $j$ . The number of edges incident to a vertex is the degree of that vertex, and by the volume of a graph we mean the sum of the degrees of its vertices. In this paper, our primary interest is in characterizing the asymptotic behavior of the eigenvalues of the random adjacency matrix as the graph size  $n$

increases. We consider the distribution of these eigenvalues over the real line and characterize this distribution through its moments sequence. Accordingly, the probability of there being an edge between nodes  $i$  and  $j$  is equal to  $\rho_n w_i^{(n)} w_j^{(n)}$ , where  $\rho_n = 1/\sum_{i=1}^n w_i^{(n)}$  is the inverse expected volume. The adjacency relations in this random graph model are represented by an  $n \times n$  real-valued, symmetric random matrix  $\mathbf{A}_n = [\mathbf{a}_{ij}^{(n)}/\sqrt{n}]$ , where  $\mathbf{a}_{ij}^{(n)}$  are independent 0-1 random variables with  $\mathbb{E}\{\mathbf{a}_{ij}^{(n)}\} = \rho_n w_i^{(n)} w_j^{(n)}$ . As we shall see in the sequel, the  $1/\sqrt{n}$  normalization is such that the distribution of the eigenvalues of the normalized adjacency matrix  $\mathbf{A}_n$  converges almost surely to a deterministic distribution that is uniquely characterized by its sequence of moments. As a main result of this paper, we explicate the technical conditions under which this convergence property holds true (cf. assumptions A1 to A4 below). Furthermore, we proffer explicit expressions for calculating this moment sequence. These expressions, in turn allow us to upper or lower bound various quantities of interest pertaining to the spectrum of the random graph adjacencies (cf. Section 4 and discussions therein).

Characterization of the convergence condition for the moments depends critically on the behavior of the extreme values of the expected degree sequence as  $n$  increases. To that end, we consider two sequences  $\{\hat{w}_n : n \in \mathbb{N}\}$  and  $\{\check{w}_n : n \in \mathbb{N}\}$  given by  $\hat{w}_n = \max_{i \in [n]} w_i^{(n)}$  and  $\check{w}_n = \min_{i \in [n]} w_i^{(n)}$  for all  $n \in \mathbb{N}$ . Another quantity of interest whose evolution with the graph size  $n$  plays an important role is the second-order average degree  $\tilde{d}_n$ , defined in [23] as,

$$\tilde{d}_n = \frac{\sum_{i=1}^n (w_i^{(n)})^2}{\sum_{i=1}^n w_i^{(n)}} = \rho_n \sum_{i=1}^n (w_i^{(n)})^2 = \rho_n \|\bar{w}_n\|_2^2.$$

For our main results to hold true, we need the expected degree sequence  $\bar{w}_n^T$  to satisfy the following assumptions:

- A1 (Sparse and Graphical):**  $\rho_n \hat{w}_n^2 < 1, \forall n$  and  $\rho_n \hat{w}_n^2 = o(1)$ .
- A2 (Logarithmic Growth):**  $\hat{w}_n/\check{w}_n = O(\log n)$ .
- A3 (Eigenvector Concentration):**  $\tilde{d}_n/\rho_n = o(n^3/\log n)$ .
- A4 (Vanishing Effect of Centralization):**  $\tilde{d}_n = O(\sqrt{n} \log n)$ .

### 2.2 Spectrum of the Chung-Lu Random Graph

In 2003, by a series of results, Chung, Lu and Vu established important asymptotic properties of the spectra of the adjacency matrices of random graph models with given expected degree sequences [23]–[25]. A key result of theirs specifies the almost sure limit of the largest eigenvalue of the adjacency matrix as follows<sup>2</sup> [23, Theorems 2.1 and 2.2].

**Theorem 1 (Largest Eigenvalue of Random Graphs).** If  $\tilde{d}_n > \sqrt{\hat{w}_n} \log n$ , then with probability one the largest eigenvalue of the (unnormalized) adjacency matrix is  $(1 + o(1))\tilde{d}_n$ ; while if  $\sqrt{\hat{w}_n} > \tilde{d}_n \log^2 n$ , then the largest eigenvalue is almost surely  $(1 + o(1))\sqrt{\hat{w}_n}$ .

2. Given three functions  $f(\cdot)$ ,  $g(\cdot)$ , and  $h(\cdot)$  we use the asymptotic notations  $f(n) = O(g(n))$  and  $f(n) = o(h(n))$  to signify the relations  $\limsup_{n \rightarrow \infty} |f(n)/g(n)| < \infty$  and  $\lim_{n \rightarrow \infty} |f(n)/h(n)| = 0$ , and we use  $f(n) \asymp (g(n))$  to mean that  $f(n) = (1 + o(1))g(n)$ .

1. Throughout this paper,  $\mathbb{R}$  and  $\mathbb{N}$  are the set of real and natural numbers,  $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ ,  $n \in \mathbb{N}$  is a parameter denoting the size of the random graph,  $[n]$  denotes  $\{1, 2, \dots, n\}$ , and  $\text{card}(\mathcal{X})$  denotes the cardinality of set  $\mathcal{X}$ . The  $n \times n$  identity matrix is denoted by  $I_n$ , random variables are printed in boldface, matrices are denoted by capital letters. Every vector is marked by a bar over its lower case letter.

Moreover, for a random graph model whose expected degree distribution obeys a power-law the largest eigenvalue is always with probability one less than  $7\sqrt{\log n} \max\{\sqrt{\hat{w}_n}, \hat{d}_n\}$ . In [23], similar conditions are established for the almost sure convergence of the  $k$ -th largest eigenvalues towards the square root of the  $k$ -th largest expected degree, and [25] also considers relevant results concerning the spectra of other matrices, such as the Laplacian. More recent results use the machinery of concentration inequalities to investigate the behavior of the graph spectra for random graphs with independent edges [36]–[38]. In [38], the author shows concentration of the spectral norm for the Laplacian and Adjacency matrices around their expectations, under certain technical conditions. These results are improved by Chung and Radcliffe [36], who use a Chernoff-type inequality and approximate the eigenvalues by those of the expected matrices. The approximation bound in [36] for the eigenvalues of the adjacency is  $O(\sqrt{\hat{w}_n} \log n)$  and it is later improved to  $(2 + o(1))\sqrt{\hat{w}_n}$  by [37].

### 3 MAIN RESULT

In this paper, we offer a moment-based characterization of the limiting distribution of the eigenvalues of the Chung-Lu random graph model given assumptions A1 to A4 on the expected degree sequence. The moment sequence provides a versatile tool in the spectral analysis of complex networks [39]–[41]. It is worth highlighting that, after normalizing the adjacency matrix by a factor  $1/\sqrt{n}$ , the largest eigenvalue escapes to infinity as  $n \rightarrow \infty$ . This is a common case when investigating the limiting distribution of a sequence of distributions that some mass escapes to infinity, which can cause the limit distribution to be not a probability distribution (does not integrate to one), in which case the underlying sequence of distribution is not “tight” (cf. [42, Section 25]). However, as we show in this paper, the tightness property holds for the sequence of spectral distributions in the Chung-Lu random graph model. It is because the mass that is escaping to infinity (finitely many largest eigenvalues) is asymptotically vanishing itself: as  $n \rightarrow \infty$ , the contribution of finitely many eigenvalue to the continuous limiting spectral distribution in the  $1/\sqrt{n}$  normalization regime is vanishingly small. By the same token, our results complement the characterization of the largest eigenvalue in [23]–[25]; which focuses on the adjacency matrix itself, as opposed to the  $(1/\sqrt{n})$ -normalized version, and investigates the growth rate and concentration of the largest eigenvalue as described in Subsection 2.2.

To describe our main results we need to introduce some terminology. Let  $\lambda_1(\mathbf{A}_n) \leq \lambda_2(\mathbf{A}_n) \leq \dots \leq \lambda_n(\mathbf{A}_n)$  be  $n$  real-valued random variables representing the  $n$  eigenvalues of the random matrix  $\mathbf{A}_n$  ordered from the smallest to the largest. We define  $\delta_x\{\cdot\}$  as the probability measure on  $\mathbb{R}$  assigning unit mass to point  $x \in \mathbb{R}$  and zero elsewhere. We also define  $\mathcal{L}_n\{\cdot\} = (1/n) \sum_{i=1}^n \delta_{\lambda_i(\mathbf{A}_n)}\{\cdot\}$  as the random probability measure on the real line that assigns a mass  $1/n$  to each one of the  $n$  eigenvalues of the random matrix  $\mathbf{A}_n$ . The corresponding distribution can be written as  $\mathbf{F}_n(x) = \mathcal{L}_n\{(-\infty, x]\} = (1/n) \text{card}(\{i \in [n] : \lambda_i(\mathbf{A}_n) \leq x\})$ , where  $\text{card}(\cdot)$  is the cardinality function, and is referred to as the *empirical spectral distribution* (ESD) for the random matrix

$\mathbf{A}_n$ . For each  $x \in \mathbb{R}$ ,  $\mathbf{F}_n(x)$  is a random variable. Moreover, we define the  $k$ -th spectral moment of the random matrix  $\mathbf{A}_n$  as the following real-valued random variable  $\mathbf{m}_k^{(n)} = (1/n) \text{trace}(\mathbf{A}_n^k) = \int_{-\infty}^{+\infty} x^k d\mathbf{F}_n(x)$ . Our main results establish the almost sure and weak convergence of the empirical spectral distribution of the (normalized) adjacency matrix  $\mathbf{A}_n$  to a deterministic distribution  $F(\cdot)$ . We call this distribution the *limiting spectral distribution* (LSD) and we characterize it through its moments sequence,  $m_k = \int_{-\infty}^{+\infty} x^k dF(x)$ .

The moments of the LSD are derived in terms of the limiting average power-sums of the normalized degree sequence, defined as follows. For each  $i \in \mathbb{N}$ , we define the limiting normalized degrees  $\sigma_i = \lim_{n \rightarrow \infty} \sqrt{\rho_n} w_i^{(n)}$ . The limiting average of the  $k$ -th power-sum of the sequence  $\{\sigma_i, i \in \mathbb{N}\}$  is given by  $\Lambda_k = \lim_{n \rightarrow \infty} (1/n) \sum_{i=1}^n \sigma_i^k$ . Note from assumption A1 that  $\sigma_i < 1$  for all  $i$ , which in turn implies that  $\Lambda_k < 1$  for all  $k$ . According to our main result, the spectral moments of the limiting spectral distribution  $F(\cdot)$  are specified as follows:

**Theorem 2 (Spectral Moments of  $\mathbf{A}_n$ ).** Consider the random graph model with given expected degree sequence  $\bar{w}_n$ , satisfying assumptions A1 to A4. With probability one,  $\mathbf{F}_n(\cdot)$  converges weakly to  $F(\cdot)$ , where  $F(\cdot)$  is the unique distribution function satisfying  $\forall k \in \mathbb{N}$ ,  $\int_{-\infty}^{+\infty} x^k dF(x) = m_k$  where for all  $s \in \mathbb{N}_0$ ,  $m_{2s+1} = 0$ , and

$$m_{2s} = \sum_{\bar{r} \in \mathcal{R}_s} \frac{2}{s+1} \binom{s+1}{r_1, \dots, r_s} \Lambda_1^{r_1} \Lambda_2^{r_2} \dots \Lambda_s^{r_s}, \quad (1)$$

with  $\bar{r} = (r_1, r_2, \dots, r_s)$ ,  $\mathcal{R}_s = \{\bar{r} \in \mathbb{N}_0^s : \sum_{j=1}^s r_j = s+1, \sum_{j=1}^s j r_j = 2s\}$ .

#### 3.1 Proof Outline

The crux of the argument is in showing that for each  $k \in \mathbb{N}$ , the  $k$ -th spectral moments  $\mathbf{m}_k^{(n)}$  converge almost surely to  $m_k$ , thence concluding by the method of moments that with probability one the empirical spectral distributions  $\mathbf{F}_n(\cdot)$  converge weakly to  $F(\cdot)$ . A centralization procedure described below will allow us to apply the results in [43], concerning random matrices with independent zero-mean entries and rank-one pattern of variances. To begin, we consider the centralized version of the adjacency  $\mathbf{A}_n$ , given by

$$\hat{\mathbf{A}}_n = \mathbf{A}_n - \mathbb{E}\{\mathbf{A}_n\} = \left[ (1/\sqrt{n}) \hat{\mathbf{a}}_{ij}^{(n)} \right]. \quad (2)$$

Note that the entries  $\hat{\mathbf{a}}_{ij}^{(n)}$  have zero means and asymptotically rank one pattern of variances given by

$$\begin{aligned} \text{Var}\{\hat{\mathbf{a}}_{ij}^{(n)}\} &= \mathbb{E}\left\{\left(\hat{\mathbf{a}}_{ij}^{(n)}\right)^2\right\} \\ &= \rho_n w_i^{(n)} w_j^{(n)} (1 - \rho_n w_i^{(n)} w_j^{(n)}) \\ &\asymp \sigma_i \sigma_j, \end{aligned} \quad (3)$$

since by assumption A1,  $\rho_n w_i^{(n)} w_j^{(n)} = o(1)$ . Furthermore, all the entries  $\hat{\mathbf{a}}_{ij}^{(n)}$  are uniformly almost surely bounded, i.e.,

$\mathbb{P}\{|\hat{\mathbf{a}}_{ij}^{(n)}| \leq 1\} = 1$ . Indeed, since the pattern of variances of the entries in  $\hat{\mathbf{A}}_n$  is (asymptotically) rank one, the random matrix  $\hat{\mathbf{A}}_n$  satisfies all the conditions necessary to apply the results in [43].

To proceed, we introduce some necessary notations. Similarly to  $\mathbf{A}_n$ , we consider the eigenvalues of  $\hat{\mathbf{A}}_n$  ordered from the smallest to the largest as  $\lambda_1(\hat{\mathbf{A}}_n) \leq \lambda_2(\hat{\mathbf{A}}_n) \leq \dots \leq \lambda_n(\hat{\mathbf{A}}_n)$  and define the random variable  $\hat{\mathbf{m}}_k^{(n)} = (1/n)\text{trace}(\hat{\mathbf{A}}_n^k)$  to be its  $k$ -th spectral moment. Also let  $\bar{\mathbf{m}}_k^{(n)} = \mathbb{E}\{\hat{\mathbf{m}}_k^{(n)}\}$  be the expected spectral moments for all  $k, n$ . The proof of our main result proceeds as follows.

Lemma 2, included in Section 7, ensures that under assumptions A3 and A4,  $\hat{\mathbf{m}}_k^{(n)} \asymp \mathbf{m}_k^{(n)}$ , almost surely for each  $k \in \mathbb{N}$ . Therefore, the effect of centralization on the spectral moments is asymptotically vanishing. Next, Lemma 4 (included in Section 7) provides asymptotically exact expressions for the expected spectral moments under assumptions A1 and A2. Lemma 3 implies that the same expressions in fact describe the almost sure limits of the spectral moments, i.e.,  $\hat{\mathbf{m}}_k^{(n)} \asymp m_k$  for all  $k \in \mathbb{N}$ . The latter, together with the earlier results, imply that under assumptions A1 to A4, i.e.,  $\mathbf{m}_k^{(n)} \asymp m_k$ , almost surely for all  $k \in \mathbb{N}$ . This pointwise almost sure convergence of the moment sequence would in turn imply the weak convergence of the empirical distributions  $\mathbf{F}_n(\cdot)$  to the deterministic distribution  $F(\cdot)$ , which is uniquely characterized by the sequence of limiting spectral moments. This fact is established in Lemma 5, completing the proof.  $\square$

### 3.2 The Case of Erdős-Rényi Random Graphs

In Erdős-Rényi random graphs, denoted by  $\mathbf{G}_{n,p}$ , each edge is realized with a probability  $p$ , independently of other edges. This is a special case of the Chung-Lu random graph model when the expected degree sequence is given by  $\bar{\mathbf{w}}_n = (np, np, \dots, np)$ . Ever since its introduction in the late 1950s by Erdős and Rényi [16], [17], properties of this well-known class of random graphs have been extensively studied and they continue to attract attention [44], [45]. Indeed, the seminal work of Füredi and Komlós [46] can be used to derive asymptotic properties of the spectra of Erdős-Rényi random graphs; with probability one, putting its largest eigenvalue at  $(1+o(1))np$  and upper-bounding the absolute values of the rest by  $(2+o(1))\sqrt{np(1-p)}$ . More recently, Feige and Ofek [47] have shown that under mild conditions on  $p$ , the largest eigenvalue of the adjacency matrix is almost surely  $pn + O(\sqrt{pn})$ , and all other eigenvalues are almost surely  $O(\sqrt{pn})$ .

Fig. 1(a) depicts the histogram of eigenvalues for a particular realization of the (unnormalized) adjacency matrix of an Erdős and Rényi graph with  $n = 1000$  nodes and  $p = 0.05$ . In particular, we can observe that the largest eigenvalue  $\lambda_1(\sqrt{n}\mathbf{A}_n)$  is located away from the remaining smaller ones. Let us also consider the centralized adjacency matrix  $\hat{\mathbf{A}}_n$  defined in (2). We plot a typical realization of the eigenvalue histogram of the unnormalized, centralized adjacency matrix  $\sqrt{n}\hat{\mathbf{A}}_n$  in Fig. 1(b). Notice that, as pointed out in Lemma 2 (proved in Section 7), the effect of centralizing the adjacency matrix  $\mathbf{A}_n(\mathbf{w})$  is to cancel out the largest eigenvalue (moving it to zero), while the bulk of eigenvalues

remains (almost) unperturbed. Subsequently, the effect of normalization on the limiting spectral distribution in the  $1/\sqrt{n}$ -normalization regime is negligible. This can be also noticed in Fig. 1(c), where we plot the empirical spectral distributions  $\mathbf{F}_n(x)$  and  $\hat{\mathbf{F}}_n(x)$ . Indeed under the  $1/\sqrt{n}$ -normalization, the largest eigenvalue of  $\mathbf{A}_n$  grows as  $\sqrt{np}$ , escaping almost surely to infinity; every other eigenvalue is almost surely  $O(\sqrt{p})$ , being asymptotically compactly supported.

Indeed, as a validity check, it is possible to rederive the limiting spectral distribution of the Erdős-Rényi random graph ensemble from Theorem 2 under assumptions A1 to A4, provided that  $p = p_n = O(\log n/\sqrt{n})$ . In particular, in this case we have that  $\hat{w}_n = \check{w}_n = np_n$ ,  $\rho_n = n^{-2}p_n^{-1}$ , and  $\tilde{d}_n = np_n$ ; therefore, the random graph ensemble satisfies assumptions A1 to A4. Thus, using Theorem 2, it is possible to verify that the asymptotic spectral moments satisfy  $m_{2s} = (p_n^s/s+1)\binom{2s}{s}$ , which correspond to the moments of a semicircular distribution supported over  $[-2\sqrt{p_n}, +2\sqrt{p_n}]$ , [48], [49]. Note that to obtain a non-trivial support for the bulk of the spectrum when  $p = p_n = O(\log n/\sqrt{n})$ , we need to investigate the LSD of the adjacency matrix under a normalization regime that is slower than  $1/\sqrt{n}$ . This, as well as other normalization regimes that pertain to sparse random graphs, have been the focus of recent interest [50]–[53].

### 4 DEGREES SPECIFIED BY RANDOM UNIFORM SAMPLING

We now consider for Chung-Lu random graphs for which the expected degree of each node  $i$  is specified as  $\mathbf{w}_i^{(n)} = f_n(\mathbf{x}_i)$ , where  $f_n(\cdot)$  are given functions with a common support normalized to be the unit interval  $[0, 1]$ , and  $\{\mathbf{x}_i, i \in \mathbb{N}\}$  is a random sample, uniformly and independently drawn from the unit interval. To illustrate our results, let us assume that  $f_n(x) = \Delta_n e^{-\alpha_n x}$ , where  $\Delta_n, \alpha_n > 0$  for all  $n$ , and  $0 \leq x \leq 1$ . Then, consider a Chung-Lu random graph with the expected degree sequence specified as  $\mathbf{w}_i^{(n)} = \Delta_n e^{-\alpha_n \mathbf{x}_i}$ ,  $i \in [n]$ . In other words, the degree distribution of the resulting graph follows an exponential distribution, which have been observed in practical scenarios, such as in structural brain networks built from diffusion imaging techniques [54], [55].

The almost sure asymptotic expression for the second-order average degree  $\tilde{d}_n$  can be obtained from a Monte Carlo average [56, Section XVI.3], resulting in the following expression:

$$\begin{aligned} \tilde{d}_n &= \frac{\sum_{i=1}^n (\mathbf{w}_i^{(n)})^2}{\sum_{i=1}^n \mathbf{w}_i^{(n)}} \asymp \frac{\int_0^1 f_n^2(x) dx}{\int_0^1 f_n(x) dx} \\ &= \frac{\int_0^1 \Delta_n^2 e^{-2\alpha_n x} dx}{\int_0^1 \Delta_n e^{-\alpha_n x} dx} = \frac{\Delta_n(1 - e^{-2\alpha_n})}{2(1 - e^{-\alpha_n})}. \end{aligned} \quad (4)$$

We know from Theorem 1 that, if  $\Delta_n > \log^2 n$ , then  $\tilde{d}_n > \sqrt{\Delta_n} \log n$ , and the largest eigenvalue is almost surely given by  $\lambda_1(\sqrt{n}\mathbf{A}_n) \asymp \tilde{d}_n$ . We now use Theorem 2 to write closed-form expressions for the asymptotic spectral moments of  $\mathbf{A}_n$ , the normalized adjacency matrix of the random graph whose expected degree sequence is given by

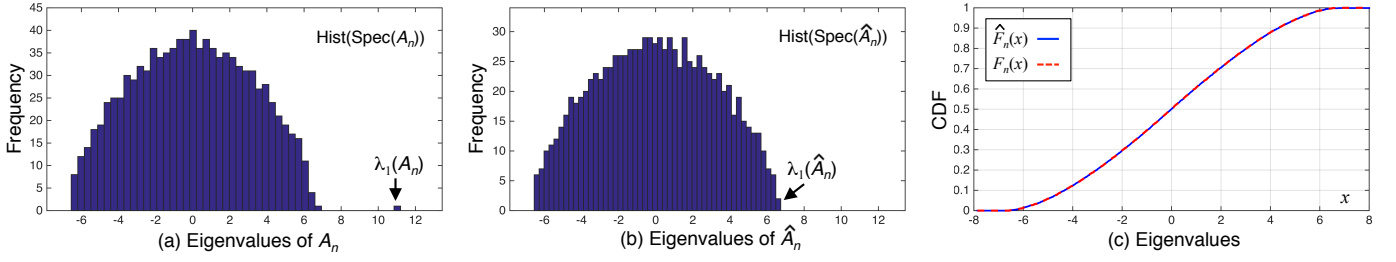


Fig. 1. (a) Eigenvalue histogram of the Erdős-Rényi random graph; (b) eigenvalue histogram of the centralized adjacency matrix ; (c) spectral distributions of the normalized adjacency matrix  $\mathbf{A}_n$  (blue solid) and the centralized version  $\hat{\mathbf{A}}_n$  (red dashed).

$\bar{\mathbf{w}}_n = (\mathbf{w}_1^{(n)}, \dots, \mathbf{w}_n^{(n)})$ . We begin by calculating the almost sure asymptotic expression of the inverse expected volume of the graph as follows,

$$\begin{aligned} \frac{1}{\rho_n} &= \sum_{i=1}^n \mathbf{w}_i^{(n)} \asymp n \int_0^1 f_n(x) dx \\ &= n \int_0^1 \Delta_n e^{-\alpha_n x} dx = \frac{n \Delta_n}{\alpha_n} (1 - e^{-\alpha_n}). \end{aligned} \quad (5)$$

We now proceed to verify the qualifying conditions for applying Theorem 2 to this random graph model. First note the following almost sure asymptotic identities for the maximum and minimum degrees,

$$\hat{\mathbf{w}}_n = \max_{i \in [n]} \mathbf{w}_i^{(n)} \asymp \Delta_n, \quad \tilde{\mathbf{w}}_n = \min_{i \in [n]} \mathbf{w}_i^{(n)} \asymp \Delta_n e^{-\alpha_n}, \quad (6)$$

both of which can be verified easily from the corresponding order statistics for the uniform distribution on the unit interval.<sup>3</sup> If we set  $\Delta_n = \sqrt{n} \log n$  and  $\alpha_n = \log \log n$ , then from the set of almost sure asymptotic identities in (4), (5) and (6) we get

$$\begin{aligned} \tilde{\mathbf{d}}_n &\asymp \Delta_n / 2 = \sqrt{n} \log n / 2 = O(\sqrt{n} \log n), \\ \frac{\hat{\mathbf{w}}_n}{\tilde{\mathbf{w}}_n} &\asymp e^{\alpha_n} = \log n = O(\log n), \\ \sqrt{\rho_n} \hat{\mathbf{w}}_n &\asymp \sqrt{\frac{\Delta_n \alpha_n}{n}} = \frac{\sqrt{\log n \log \log n}}{n^{1/4}} = o(1), \\ \frac{\tilde{\mathbf{d}}_n}{\rho_n} &\asymp \frac{n \Delta_n^2}{2 \alpha_n} = \frac{n^2 \log^2 n}{2 \log \log n} = o\left(\frac{n^3}{\log n}\right), \end{aligned}$$

Hence, assumptions A1 to A4 are all satisfied and we can apply Theorem 2 to obtain the closed-form expressions for the limiting spectral moments of the normalized adjacency matrix  $\mathbf{A}_n$ . First note that the  $k$ -th order limiting averages  $\Lambda_k$  are almost surely given by

$$\Lambda_k \asymp \int_0^1 (\sqrt{\rho_n} f_n(x))^k dx = \int_0^1 (\sqrt{\rho_n} \Delta_n e^{-\alpha_n x})^k dx,$$

3. More specifically, if we note that  $\min_{i \in [n]} \mathbf{x}_i$  and  $\max_{i \in [n]} \mathbf{x}_i$  are respectively distributed as  $Beta(1, n)$  and  $Beta(n, 1)$  variables [57, Chapter 2], then the claimed almost sure limits follow by the Borel-Cantelli lemma. To see how, consider their expected values:  $\mathbb{E}\{\min_{i \in [n]} \mathbf{x}_i\} = 1/(n+1)$  and  $\mathbb{E}\{\max_{i \in [n]} \mathbf{x}_i\} = n/(n+1)$ , and apply the Chebyshev inequality to their quadratically decaying common variance,  $\text{Var}\{\max_{i \in [n]} \mathbf{x}_i\} = \text{Var}\{\min_{i \in [n]} \mathbf{x}_i\} = n/((n+1)^2(n+2))$ .

which results in:

$$\Lambda_k = \frac{(\sqrt{\rho_n} \Delta_n)^k}{k \alpha_n} (1 - e^{-k \alpha_n}).$$

Under these conditions, we can apply Theorem 2 to obtain the following closed-form expression for the asymptotic spectral moments of  $\mathbf{A}_n$ :

$$\begin{aligned} m_{2s} &\asymp \sum_{\bar{r} \in \mathcal{R}_s} \frac{2}{s+1} \binom{s+1}{r_1, \dots, r_s} \prod_{k=1}^s \frac{(\sqrt{\rho_n} \Delta_n)^{k r_k}}{(k \alpha_n)^{r_k}} \\ &= \frac{\rho_n \Delta_n^{2s}}{\alpha_n^{s+1}} \sum_{\bar{r} \in \mathcal{R}_s} \frac{2}{s+1} \binom{s+1}{r_1, \dots, r_s} \prod_{k=1}^s k^{-r_k} \\ &= \frac{\log^s n}{n^{s/2} \log \log n} \sum_{\bar{r} \in \mathcal{R}_s} \frac{2}{s+1} \binom{s+1}{r_1, \dots, r_s} \prod_{k=1}^s k^{-r_k}, \quad (7) \end{aligned}$$

where in the second equality we have used the identities  $\sum_{k=1}^s r_k = s+1$  and  $\sum_{k=1}^s k r_k = 2s$ . The histogram of the eigenvalues for the normalized adjacency matrix  $\mathbf{A}_n$  of a particular realization with parameters  $\Delta = 10$ ,  $\alpha = 1$ , and  $n = 1000$  is plotted in Fig. 2. The largest eigenvalue of the centralized adjacency matrix  $\hat{\mathbf{A}}_n = \mathbf{A}_n - [\sqrt{\rho_n} \mathbf{w}_i^{(n)} \mathbf{w}_j^{(n)}]_{i,j=1}^n$  in this realization is given by  $\lambda_n(\hat{\mathbf{A}}_n) = 5.6214$ . We can upper and lower bound this eigenvalue using the  $k$ -th spectral moment, for any  $k$ , as follows [27, Equation (2.66)]:

$$\text{trace}(\hat{\mathbf{A}}_n^k)^{1/k} \leq \lambda_n(\hat{\mathbf{A}}_n) \leq (n \cdot \text{trace}(\hat{\mathbf{A}}_n^k))^{1/k}. \quad (8)$$

If we consider  $k = 20$  in (8) and use the asymptotic spectral moment  $m_{20}$  available from (7) to replace for  $\text{trace}(\hat{\mathbf{A}}_n^{20})$ , then the lower and upper bounds on  $\lambda_n(\hat{\mathbf{A}}_n)$  are given by:  $(m_{20})^{1/20} = 4.3193$  and  $(n \cdot m_{20})^{1/20} = 6.1011$ . These values compare reasonably with the empirically observed value  $\lambda_n(\hat{\mathbf{A}}_n) = 5.6214$ . Furthermore, using the techniques proposed in [40], we can formulate semi-definite programs that improve the bounds in (8) by taking into account the knowledge of all spectral moments up to a fixed order, as described in [43, Section 3].

## 5 THE CASE OF POWER-LAW RANDOM GRAPHS

In this section, we study the eigenvalue distribution of the random power-law graph proposed by Chung et al. in [23]. This random graph presents a expected degree sequence given by  $\bar{w}_n = (w_1^{(n)}, w_2^{(n)}, \dots, w_n^{(n)})$  such that  $w_i^{(n)} = c i^{-1/\beta-1}$ , for  $i = i_0 + 1, \dots, i_0 + n$ , where  $\beta$  is the exponent of the power-law degree distribution. In this

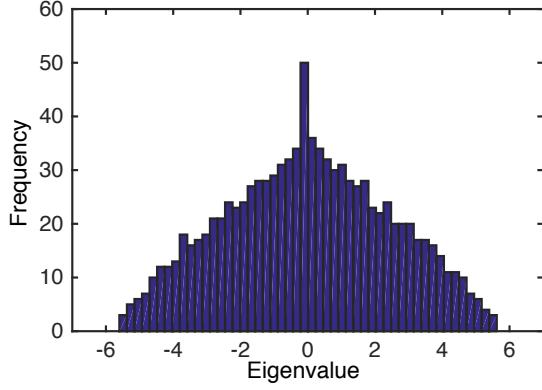


Fig. 2. Eigenvalue histogram of a sample realization from the random graph model with exponential degree sequence and  $n = 1000$  vertices.

model, we can prescribe a maximum and average expected degrees, denoted by  $\Delta$  and  $d$ , respectively, by choosing the following values of  $c$  and  $i_0$  [23]:

$$c = \frac{\beta - 2}{\beta - 1} d n^{\frac{1}{\beta-1}}, \quad i_0 = n \left( \frac{d(\beta - 2)}{\Delta(\beta - 1)} \right)^{\beta-1}.$$

For power-law degree distributions, we can asymptotically evaluate the averaged  $k$ -th power-sums of the expected degrees as  $n \rightarrow \infty$ :

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n w_i^k &= \frac{1}{n} \sum_{i=i_0}^{i_0+n-1} \left( c i^{-\frac{1}{\beta-1}} \right)^k \\ &\asymp \frac{1}{n} \int_{i_0}^{i_0+n} \left( c x^{-\frac{1}{\beta-1}} \right)^k dx \\ &= \frac{1}{n} c^k \frac{\beta - 1}{\beta - 1 - k} x^{\frac{\beta-1-k}{\beta-1}} \Big|_{i_0}^{i_0+n} \\ &= d^k f\left(\frac{d}{\Delta}; \beta, k\right), \end{aligned}$$

where

$$\begin{aligned} f\left(\frac{d}{\Delta}; \beta, k\right) &= \left(\frac{\beta - 2}{\beta - 1}\right)^k \frac{\beta - 1}{\beta - 1 - k} \times \dots \\ &\left[ \left( \left( \frac{d(\beta - 2)}{\Delta(\beta - 1)} \right)^{\beta-1} + 1 \right)^{\frac{\beta-1-k}{\beta-1}} - \left( \frac{d(\beta - 2)}{\Delta(\beta - 1)} \right)^{\beta-1-k} \right]. \end{aligned} \quad (9)$$

Notice that when the largest degree is much larger than the average degree, i.e.,  $\Delta = \omega(d)$ , the expression inside the square brackets in (9) tends to one; hence (9) simplifies as follows:

$$f\left(\frac{d}{\Delta}; \beta, k\right) = (1 + o(1)) \left(\frac{\beta - 2}{\beta - 1}\right)^k \frac{\beta - 1}{\beta - 1 - k} = \tilde{f}(\beta, k).$$

Therefore, for  $k = 1$ , we have that  $f\left(\frac{d}{\Delta}; \beta, k\right) \asymp 1$  and  $\frac{1}{n} \sum_{i=1}^n w_i \asymp d$ , the expected average degree. Furthermore, the above expressions can be used to compute the normalized power-sums in Theorem 2,  $\Lambda_k = \lim_{n \rightarrow \infty} (1/n) \rho_n^{k/2} \sum_{i=1}^n \left( w_i^{(n)} \right)^k$ , which can then be used to compute closed-form values for the asymptotic expected spectral moments using (1).

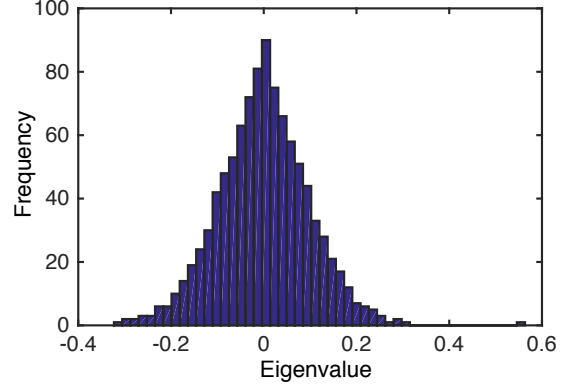


Fig. 3. Eigenvalue histogram for a power-law random graph.

Order	2	4	6	8
Theoretical	9.56e-3	3.10e-4	1.81e-5	1.46e-6
Empirical	9.5e-3	2.89e-4	1.62e-5	1.26e-6

TABLE 1  
Theoretical versus empirical spectral moments  
for the power law in Example 1

**Example 1.** Numerical verification of the asymptotic spectral moments for the power-law degree distributions. In the following numerical simulations, we verify the validity of Theorem 2 by comparing the empirical average for the first five even-order spectral moments of a power-law random graph with  $n = 1000$ ,  $\beta = 3$ ,  $\Delta = 100$ , and  $d = 10$ . The eigenvalue histogram of one particular realization is plotted in Fig. 3. In Table 1, we compare the theoretical values of the even spectral moments of the centralized adjacency matrix with the empirical values of a random sample. We would like to remark how, as reported in [31], [32] the empirical spectral distribution of the power-law under consideration resembles a “triangular” law. In the sequel we use Theorem 2 to study if this distribution is in fact a triangle.

### 5.0.1 The Triangular Spectrum

Many empirical studies of real-world networks have reported triangle-like eigenvalue spectra [31], [32]. In what follows, we want to compare the spectral density of the power-law graph with the triangular density function, given by:

$$t(x; b) = \begin{cases} \frac{2}{b} + \frac{4}{b^2}x, & \text{for } x \in [-b/2, 0], \\ \frac{2}{b} - \frac{4}{b^2}x, & \text{for } x \in (0, b/2], \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

The moments of this density function can be calculated as follows:

$$\begin{aligned} \hat{m}_k(b) &= \int_{-b/2}^{b/2} x^k t(x; b) dx \\ &= \left(1 + (-1)^k\right) \frac{(b/2)^k}{(k+1)(k+2)}. \end{aligned} \quad (11)$$

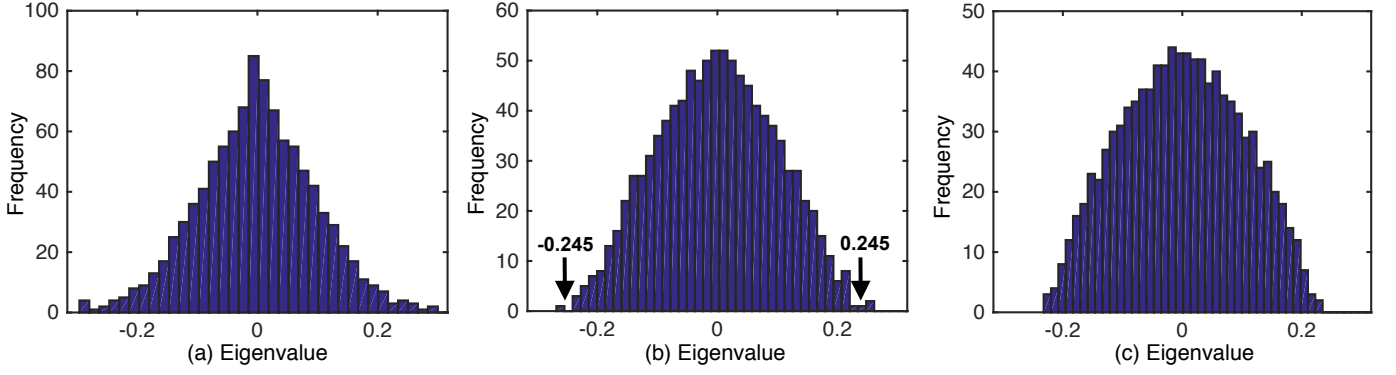


Fig. 4. Histograms of the eigenvalues for power-law networks with  $n = 1000$ ,  $\Delta = 100$ ,  $d = 10$ , and the following values of  $\beta$ : (a)  $\beta = 3$ , (b)  $\beta = \beta_\Delta \approx 4.44$ , and (c)  $\beta = 6$ .

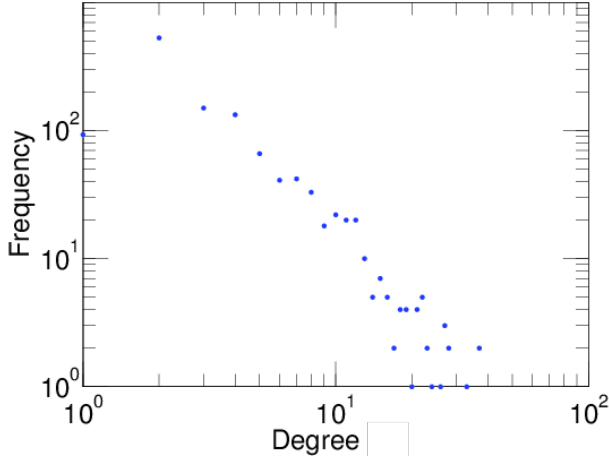


Fig. 5. Power-law degree distribution of the air traffic control network.

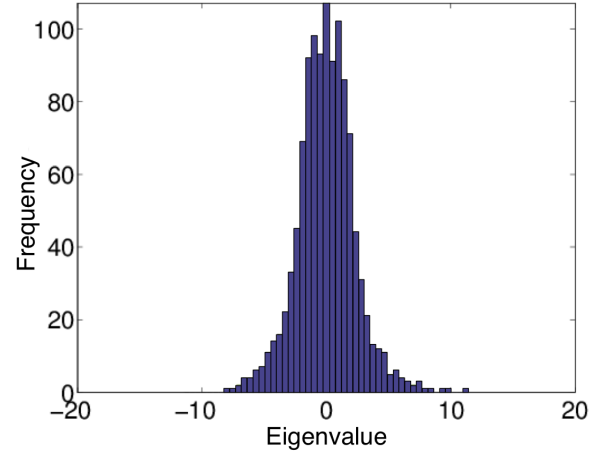


Fig. 6. Eigenvalue histogram of the air traffic control network.

In what follows, we want to find conditions under which the spectral density resembles a triangular distribution. We measure the similarity in terms of the moments, in particular, in terms of the kurtosis<sup>4</sup> of the distribution. From (11), the second and fourth moments of the triangular law are given by  $\hat{m}_2(b) = \frac{b^2}{24}$  and  $\hat{m}_4(b) = \frac{b^4}{240}$ . The kurtosis of the triangular distribution is then given by

$$\hat{\kappa} = \frac{\hat{m}_4(b)}{(\hat{m}_2(b))^2} = \frac{12}{5}. \quad (12)$$

On the other hand, from Theorem 2, we can compute the kurtosis of the power-law network:

$$\kappa = \frac{m_4}{m_2^2} = \frac{2\Lambda_1^2\Lambda_2}{(\Lambda_1^2)^2} = \frac{2\Lambda_2}{\Lambda_1^2} = \frac{2\frac{\tilde{d}}{n}}{\frac{1}{n^2\rho}} = 2n\rho\tilde{d}. \quad (13)$$

In our analysis, we use the difference between kurtoses as a measure of how far the spectral distribution is from the triangular law. Therefore, the spectral distribution that is closest to the triangular law is the one for which  $\kappa = \hat{\kappa}$ .

4. The kurtosis of a distribution is defined as  $\kappa = \mu_4/\sigma^4$ , where  $\mu_4$  is the fourth moment about the mean and  $\sigma$  is the standard deviation. This ratio is a measurements of how heavy-tailed the distribution is.

According to (12) and (13),  $\kappa = \hat{\kappa}$  is satisfied when

$$n\rho\tilde{d} = \frac{6}{5}.$$

Furthermore, if  $\kappa < 12/5$  (respectively,  $\kappa > 12/5$ ), then the tail of the spectral distribution is “fatter” (respectively, “thinner”) than the tail of the triangular law (see Fig. 4).

Moreover, since  $\rho \asymp \frac{1}{n\tilde{d}}$  and  $\tilde{d} \asymp d\left(\frac{\beta-2}{\beta-1}\right)^{\frac{\beta-1}{\beta-3}}$  (when  $\Delta = \omega(d)$ ), we have that  $\kappa = \hat{\kappa}$  when

$$\frac{(\beta-2)^2}{(\beta-1)(\beta-3)} = \frac{6}{5}.$$

Solving the above equation, we obtain the following critical value of  $\beta$  for which  $\kappa$  matches  $\hat{\kappa}$ :  $\beta_\Delta = 2 + \sqrt{6} \approx 4.44$ . In other words, the kurtosis of the spectral distribution is equal to that of the triangular law only when  $\beta = \beta_\Delta$ .

**Remark 1.** In practice, the values of  $\beta$  in real world networks are below the threshold value  $\beta^* \approx 4.44$  (cf. [58]). Therefore, the spectral tails of real networks are mostly “fat”, which is in accordance with empirical observations reported in [13]. For example, we include in Fig. 5 the power-law degree distribution of an air traffic control network constructed from the USA’s FAA (Federal Aviation Administration) National Flight Data Center (NFDC), Preferred Routes Database [59]. Nodes



in this network represent airports and links are created from strings of preferred routes recommended by the NFDC. This network has a total of 1,226 nodes, 2,615 edges, and a power-law exponent of 3.7 (which is below the threshold value  $\beta_\Delta$ ). As predicted, the corresponding eigenvalue distribution, plotted in Fig. 6, presents a ‘fat’ tail.

When the spectral distribution resembles a triangular law, it is possible to approximate the support of the spectral bulk of eigenvalues by computing the value of  $b$  in (10) such that the second moment of the triangular distribution matches the second moment of the theoretical spectral distribution. Since the second moment of the spectral distribution is given by  $m_2 = \Lambda_1^2 = \frac{1}{n^2\rho}$  and the second moment of the triangular distribution is  $\hat{m}_2 = \frac{b^2}{24}$ , the value of  $b$  for which these moments match is given by:  $b_\Delta = \frac{1}{n}\sqrt{\frac{24}{\rho}}$ . In Fig. 4(b), we indicate this value of  $b_\Delta$  in the x-axis, which in this case is given by  $b_\Delta \approx 0.245$ .

One may ask if the spectral distribution of a random power-law network does indeed follow a triangular density when  $\beta = \beta_\Delta$ ? The answer is no. This can be verified by comparing the sixth moments of the triangular distribution with  $b = b_\Delta$  with the sixth moment of the power-law spectral density when  $\beta = \beta_\Delta$ . In particular, the sixth moment of the triangular law is given by

$$\hat{m}_6(b_\Delta) = \frac{54}{7} \frac{1}{n^6\rho^3} \approx \frac{7.71}{n^6\rho^3}.$$

On the other hand, from Theorem 2, the sixth spectral moment of the power-law network is given by

$$m_6(\beta_\Delta) = 2\Lambda_1^3\Lambda_3 + 3\Lambda_1^2\Lambda_2^2.$$

Since  $\Lambda_1 = \frac{1}{n\sqrt{\rho}}$ ,  $\Lambda_2 = \frac{1}{n^2\rho} \left( \frac{\beta_\Delta-2}{\beta_\Delta-1} \right)^2 \frac{\beta_\Delta-1}{\beta_\Delta-3} = \frac{1}{n^2\rho} \frac{6}{5}$ , and  $\Lambda_3 = \frac{1}{n^3\rho^{3/2}} \frac{6^{3/2}}{(1+\sqrt{6})^2(-2+\sqrt{6})}$ , we have that

$$m_6(\beta_\Delta) = \frac{2}{n^6\rho^3} \left[ \frac{6}{25} (27 + \sqrt{6}) \right] \approx \frac{14.14}{n^6\rho^3},$$

which does not coincide with  $\hat{m}_6(b_\Delta)$ .

## 6 CONCLUSIONS

In this work, we have investigated the asymptotic behavior of the bulk of eigenvalues of the adjacency matrix of random graph with specified expected degrees sequence. We have showed that, in the  $1/\sqrt{n}$  normalization regime and under some technical assumptions on the expected degrees sequence, the empirical spectral distribution of the adjacency matrix converges weakly to a deterministic distribution, which we have characterized by providing closed-form expressions for its limiting spectral moments.

We have illustrated the application of our results by analyzing the spectral distribution of large-scale networks with exponential degree distributions, which appear in structural brain networks obtained from diffusion imaging. We have also applied our results to analyze the spectrum of power-law random graphs, which are of great practical importance. Using the closed-form expressions for the asymptotic spectral moments in Theorem 2, we have investigated the triangle-like spectrum of power-law random

graphs. In particular, we have provided quantitative relationships to show how the parameters of the power-law degree distribution affect the shape and properties of the graph spectrum. Furthermore, closed-form expressions of the asymptotic spectral moments allow us to bound spectral properties of practical interest, such as the support of the spectral bulk.

## 7 PROOF OF MAIN RESULTS

The argument leading to the proof of our main convergence result in Theorem 2 is executed in three steps. We begin by showing (in Section 7.1, below) that with probability one  $\lim_{n \rightarrow \infty} \mathbf{m}_k^{(n)} = \lim_{n \rightarrow \infty} \hat{\mathbf{m}}_k^{(n)}$ , i.e., the adjacency matrix  $\mathbf{A}_n$  and its centralized version  $\hat{\mathbf{A}}_n$  share the same almost sure limits for their spectral moments. Next, in Section 7.2, we refine the arguments used in [43] to prove the almost sure limits  $\lim_{n \rightarrow \infty} \hat{\mathbf{m}}_k^{(n)} = m_k$ , with  $m_k$  given by (1) in Theorem 2. Finally, in Section 7.3, we apply the method of moments to conclude the weak and almost sure convergence of the empirical spectral distributions  $\mathbf{F}_n(\cdot)$  to the deterministic distribution  $F(\cdot)$ , which is characterized by the moments sequence  $m_k, k \in \mathbb{N}$ .

### 7.1 Effect of Centralization on the Spectral Moments

The following set of results measures the effect of the centralization in (2) by comparing the spectral moments of  $\hat{\mathbf{A}}_n$  and  $\mathbf{A}_n$  as  $n \rightarrow \infty$ . Indeed, centralization by subtracting the mean  $\mathbb{E}\{\mathbf{A}_n\}$  from the adjacency matrix  $\mathbf{A}_n$  has the effect of shifting the largest eigenvalue towards zero. Example 1 demonstrates this shifting; however, in the sequel we shall show that the subsequent effect on the spectral moments is asymptotically vanishing, under certain mild assumptions on the degree sequences.

We know from Theorem 1 that, if  $\tilde{d}_n > \sqrt{\tilde{w}_n} \log n$ , then  $\lambda_1(\mathbf{A}_n) \asymp \tilde{d}_n$  with probability one. We use a variation of this result to prove that the column vector of expected degrees, denoted by  $\bar{w}_n^T = (w_1^{(n)}, \dots, w_n^{(n)})^T$ , is asymptotically almost surely an eigenvector of  $\mathbf{A}_n$  associated its largest eigenvalue. In particular, as  $\lambda_n(\mathbf{A}_n)$  concentrates around  $\tilde{d}_n$ , the vector  $(1/n)\mathbf{A}_n\bar{w}_n$  also concentrates around the vector  $\frac{1}{n\sqrt{\tilde{d}_n}}\tilde{d}_n\bar{w}_n$ . This is important when characterizing the effect of the centralization in (2) in light of the fact that

$$\begin{aligned} \mathbb{E}\{\mathbf{A}_n\} &= \left[ \frac{\rho_n}{\sqrt{n}} w_i w_j \right]_{i,j \in [n]} \\ &= \frac{\rho_n}{\sqrt{n}} \bar{w}_n \bar{w}_n^T \\ &= \frac{\tilde{d}_n}{\sqrt{n} \|\bar{w}_n\|_2^2} \bar{w}_n \bar{w}_n^T. \end{aligned} \quad (14)$$

**Lemma 1 (Eigenvector Concentration).** Under assumption A3, it is true that  $\frac{1}{n}\mathbf{A}_n\bar{w}_n \asymp \frac{1}{n\sqrt{\tilde{d}_n}}\tilde{d}_n\bar{w}_n$  almost surely.

*Proof:* The  $i$ -th component of  $\mathbf{A}_n\bar{w}_n$  is a random variable given by:

$$[\mathbf{A}_n\bar{w}_n]_i = \sum_{j=1}^n \frac{\mathbf{a}_{ij}^{(n)}}{\sqrt{n}} w_j^{(n)}. \quad (15)$$



Since  $\mathbf{a}_{ij}^{(n)}$  is a Bernoulli random variable with  $\mathbb{P}\{\mathbf{a}_{ij}^{(n)} = 1\} = \rho_n w_i^{(n)} w_j^{(n)}$ , we have that

$$\mathbb{E}\{[\mathbf{A}_n \bar{w}_n]_i\} = \frac{w_i^{(n)}}{\sqrt{n}} \sum_{j=1}^n \rho_n (w_j^{(n)})^2 = \frac{\tilde{d}_n}{\sqrt{n}} w_i^{(n)},$$

and  $\mathbb{E}\{\mathbf{A}_n \bar{w}_n\} = (\tilde{d}_n/\sqrt{n})\bar{w}_n$ . Next, note that each of the summands  $\mathbf{a}_{ij}^{(n)} w_j^{(n)}/\sqrt{n}$  in (15) are independent bounded random variables satisfying  $\mathbf{a}_{ij}^{(n)} w_j^{(n)}/\sqrt{n} \in [0, w_j^{(n)}/\sqrt{n}]$  almost surely. Hence, we can apply Hoeffding's inequality [60, Theorem 2] to obtain that for each  $i$  and any  $\varepsilon > 0$

$$\begin{aligned} & \mathbb{P}\left\{\frac{1}{n} |[\mathbf{A}_n \bar{w}_n]_i - \frac{\tilde{d}_n}{\sqrt{n}} w_i^{(n)}| \geq \varepsilon\right\} \\ & \leq 2 \exp\left(\frac{-2n^3 \varepsilon^2}{\sum_{i=1}^n (w_i^{(n)})^2}\right) \\ & = 2e^{-2n^3 \rho_n \varepsilon^2 / \tilde{d}_n}. \end{aligned}$$

Next, note that given  $\tilde{d}_n/\rho_n = o(n^3/\log n)$  per assumption A3, with  $\varepsilon$  as in above and for any  $\alpha > 1$ , we get that when  $n$  is large enough,  $\tilde{d}_n/\rho_n < 2n^3 \varepsilon^2/(\alpha \log n)$ ; whence  $2e^{-2n^3 \rho_n \varepsilon^2 / \tilde{d}_n} < 2/n^\alpha$  forms a summable series in  $n$ , and by the Borel-Cantelli lemma [42, Theorem 4.3] we get that

$$\mathbb{P}\left\{\left|\frac{1}{n} [\mathbf{A}_n \bar{w}_n]_i - \frac{\tilde{d}_n}{n\sqrt{n}} w_i^{(n)}\right| \geq \varepsilon, \text{ infinitely often}\right\} = 0,$$

which holds true for any  $\varepsilon > 0$ , and therefore we have

$$\mathbb{P}\left\{\lim_{n \rightarrow \infty} \frac{1}{n} [\mathbf{A}_n \bar{w}_n]_i = \lim_{n \rightarrow \infty} \frac{\tilde{d}_n}{n\sqrt{n}} w_i^{(n)}\right\} = 1.$$

The claimed concentration of eigenvector around  $\bar{w}_n$  now follows by the countable intersections of the above almost sure events over all  $i \in \mathbb{N}$ .  $\square$

We can now proceed to give conditions under which the spectral moments of  $\mathbf{A}_n$  and  $\hat{\mathbf{A}}_n$  are asymptotically almost surely identical, and therefore the effect of centralization on spectral moments is asymptotically vanishing.

**Lemma 2 (Vanishing Effect of Centralization).** Under assumptions A3 and A4, it is true that  $\hat{\mathbf{m}}_k^{(n)} \asymp \mathbf{m}_k^{(n)}$ , almost surely, for each  $k \in \mathbb{N}$ .

*Proof:* To begin, consider the  $k = 1$  case. From (2), we have

$$\hat{\mathbf{m}}_1^{(n)} = \frac{1}{n} \text{trace}(\hat{\mathbf{A}}_n) = \frac{1}{n} \text{trace}(\mathbf{A}_n - \mathbb{E}\{\mathbf{A}_n\}).$$

From (14) we know that

$$\text{trace}(\mathbb{E}\{\mathbf{A}_n\}) = \frac{\tilde{d}_n}{\sqrt{n} \|\bar{w}_n\|_2^2} \text{trace}(\bar{w}_n \bar{w}_n^T) = \frac{\tilde{d}_n}{\sqrt{n}},$$

wherefrom it follows that  $\text{trace}(\hat{\mathbf{A}}_n) = \text{trace}(\mathbf{A}_n) - \tilde{d}_n/\sqrt{n}$  is true for all  $n$ , and in particular with probability one as  $n \rightarrow \infty$ . For general  $k \in \mathbb{N}$ , we have

$$\hat{\mathbf{m}}_k^{(n)} = \frac{1}{n} \text{trace}(\hat{\mathbf{A}}_n^k) = \frac{1}{n} \text{trace}(\mathbf{A}_n - \mathbb{E}\{\mathbf{A}_n\})^k.$$

To proceed, consider the binomial expansion of  $(\mathbf{A}_n - \mathbb{E}\{\mathbf{A}_n\})^k$  consisting of a sum of the product of non-commutative elements as follows:

$$\begin{aligned} & \mathbf{A}_n^k + \mathbf{A}_n^{k-1}(-\mathbb{E}\{\mathbf{A}_n\}) + \mathbf{A}_n^{k-2}(-\mathbb{E}\{\mathbf{A}_n\})\mathbf{A}_n + \dots \\ & + \mathbf{A}_n^{k-2}(-\mathbb{E}\{\mathbf{A}_n\})^2 + \mathbf{A}_n^{k-3}(-\mathbb{E}\{\mathbf{A}_n\})\mathbf{A}_n(-\mathbb{E}\{\mathbf{A}_n\}) + \dots \end{aligned}$$

Consider any product term of the form,

$$\Pi(\bar{k}) = \mathbf{A}_n^{k_1}(-\mathbb{E}\{\mathbf{A}_n\})^{k_2} \mathbf{A}_n^{k_3} \dots (-\mathbb{E}\{\mathbf{A}_n\})^{k_p}, \quad (16)$$

where  $\bar{k} = (k_1, \dots, k_p)$  is an integer partition of  $k$ , consisting of  $p \geq 1$  positive integers satisfying  $k_1 + \dots + k_{p-1} = k$ . Let  $\tilde{k}$  be the sum of evenly indexed integers, i.e.,  $\tilde{k} = k_2 + k_4 + \dots + k_{2 \cdot \lfloor p/2 \rfloor}$ . We claim that as  $n \rightarrow \infty$ ,

$$\frac{1}{n} \text{trace}(\Pi(\bar{k})) \rightarrow \frac{\tilde{d}_n^{\tilde{k}}}{n^{(k/2)+1}} (-1)^{\tilde{k}}, \quad (17)$$

almost surely. To see why, first note that for any  $k_i$ ,  $(\bar{w}_n \bar{w}_n^T)^{k_i} = \|\bar{w}_n\|_2^{2(k_i-1)} \bar{w}_n \bar{w}_n^T$ , so taking the  $k_i$ -th power of both sides in (14), we get

$$(-\mathbb{E}\{\mathbf{A}_n\})^{k_i} = \frac{(-\tilde{d}_n)^{k_i}}{n^{k_i/2} \|\bar{w}_n\|_2^2} \bar{w}_n \bar{w}_n^T,$$

and replacing in (16) yields

$$\Pi(\bar{k}) = \frac{(-\tilde{d}_n)^{\tilde{k}}}{n^{\tilde{k}/2}} \mathbf{A}_n^{k_1} \frac{\bar{w}_n \bar{w}_n^T}{\|\bar{w}_n\|_2^2} \mathbf{A}_n^{k_3} \dots \frac{\bar{w}_n \bar{w}_n^T}{\|\bar{w}_n\|_2^2}. \quad (18)$$

Under assumption A3, from Lemma 1 we know that  $(1/n)\mathbf{A}_n \bar{w}_n \asymp (\tilde{d}_n/n^{3/2})\bar{w}_n$  almost surely. Multiplying both sides by  $\mathbf{A}_n^{k_i-1}$  by any integer  $k_i$  yields

$$\frac{1}{n} \mathbf{A}_n^{k_i} \bar{w}_n \asymp \frac{\tilde{d}_n^{k_i}}{n^{(k_i/2)+1}} \bar{w}_n,$$

almost surely. Hence, taking the limits of both sides in (18) we get

$$\frac{1}{n} \Pi(\bar{k}) \asymp (-1)^{\tilde{k}} \frac{\tilde{d}_n^{\tilde{k}}}{n^{(k/2)+1}} \frac{\bar{w}_n \bar{w}_n^T}{\|\bar{w}_n\|_2^2},$$

almost surely. Taking the trace of both sides and using the fact that  $\text{trace}(\bar{w}_n \bar{w}_n^T / \|\bar{w}_n\|_2^2) = 1$ , leads to (17) as claimed. The above applies invariably to any of the product terms appearing in the binomial expansion of  $(\mathbf{A}_n - \mathbb{E}\{\mathbf{A}_n\})^k$ , with the exception of the leading term,  $\text{trace}(\mathbf{A}_n^k)$ , which does not simplify any further. Next, note that given any  $1 \leq \tilde{k} \leq k$ , the number of terms  $\Pi(\bar{k})$  in the binomial expansion for which (17) holds true, is exactly  $\binom{k}{\tilde{k}}$ ; wherefore we get that with probability one,

$$\begin{aligned} \text{trace}(\hat{\mathbf{A}}_n^k) &= \frac{1}{n} \text{trace}(\mathbf{A}_n - \mathbb{E}\{\mathbf{A}_n\})^k \\ &\asymp \frac{1}{n} \left( \text{trace}(\mathbf{A}_n^k) + \frac{\tilde{d}_n^k}{n^{k/2}} \sum_{\tilde{k}=1}^k (-1)^{\tilde{k}} \binom{k}{\tilde{k}} \right). \end{aligned}$$

We next use a trite identity,  $0 = (1-1)^k = 1 + \sum_{\tilde{k}=1}^k (-1)^{\tilde{k}} \binom{k}{\tilde{k}}$  to get

$$\frac{1}{n} \text{trace}(\hat{\mathbf{A}}_n^k) \asymp \frac{1}{n} \left( \text{trace}(\mathbf{A}_n^k) - \frac{\tilde{d}_n^k}{n^{k/2}} \right),$$

almost surely. The claim now follows upon noting that  $\frac{1}{n}(\tilde{d}_n/\sqrt{n})^k = o(1), \forall k \in \mathbb{N}$ , which is true because  $\tilde{d}_n = O(\sqrt{n \log n})$  per assumption A4.  $\square$

## 7.2 Almost Sure Limits of the Spectral Moments

In this section, we establish that the asymptotic relations for the almost sure limits of the spectral moments of the centralized adjacency matrix  $\hat{\mathbf{A}}_n$  are indeed given by (1) in Theorem 2, i.e. we have  $\hat{\mathbf{m}}_k^{(n)} \asymp m_k$ , almost surely for each  $k \in \mathbb{N}$ . The underlying arguments parallel those in [43] and some details are omitted here for brevity. In particular, Lemma A3.12 in [43] on the almost sure convergence of the spectral moments holds true in the case of our random matrix ensemble  $\hat{\mathbf{A}}_n$  and with no change to the proof steps therein. Accordingly, we have:

**Lemma 3 (Almost Sure Convergence of the Spectral Moments).** For any  $k \in \mathbb{N}$ , if  $\bar{m}_k^{(n)} \asymp m_k$ , it is true that  $\hat{\mathbf{m}}_k^{(n)} \asymp m_k$  almost surely.

Hence, for investigating the almost sure convergence of the spectral moments of  $\hat{\mathbf{A}}_n$ , it suffices to consider their expectations in the limit. This is achieved by first identifying the terms that asymptotically dominate the behavior of each moment of any order, as in [43, Theorem 2.7], and then deriving the asymptotically exact expressions for each of the identified terms following [43, Subsection 2.4]. The analysis leading to Theorem 2.7 and Theorem 2.8 of [43] holds true verbatim in the case of  $\hat{\mathbf{A}}_n$ , provided that we replace  $\hat{\sigma}_n$  and  $\hat{\sigma}_n$  therein, by  $\sqrt{\rho_n} \hat{w}_n$  and  $\sqrt{\rho_n} \hat{w}_n$  respectively. By the same token, Theorem 2.9 in [43] can be recycled for our random matrix ensemble  $\hat{\mathbf{A}}_n$ , establishing (1) as the asymptotically exact expressions for the even moments. Here, in addition to replacing  $\hat{\sigma}_n$  and  $\hat{\sigma}_n$ , we should also change every  $\sigma_i$  term in that proof to  $\sqrt{\rho_n} w_i^{(n)}$  for finite  $n$ ; however, this adjustment does not influence the conclusion of the theorem which concerns the limit as  $n \rightarrow \infty$  of the dominant term identified by Theorem 2.7, since we have defined  $\sigma_i = \lim_{n \rightarrow \infty} \sqrt{\rho_n} w_i^{(n)}, \forall i$  and per (3),  $\mathbb{E} \left\{ \left( \hat{\mathbf{a}}_{ij}^{(n)} \right)^2 \right\} \asymp \sigma_i \sigma_j$ . Thereby, combining the results of Theorems 2.7, 2.8 and 2.9 in [43], and after the minor adjustments indicated above, we obtain:

**Lemma 4 (Limiting Spectral Moments).** Under assumptions A1 and A2, it is true that  $\bar{m}_{2s}^{(n)} \asymp m_{2s}$  and  $\bar{m}_{2s-1}^{(n)} = o(1)$  for all  $s \in \mathbb{N}$ .

## 7.3 Weak Convergence of the Spectral Distribution

The results in Sections 7.1 and 7.2 enable us to claim that, under assumptions A1 to A4, the spectral moments sequence  $\{\mathbf{m}_k^{(n)}, k \in \mathbb{N}\}$  converges pointwise almost surely to the deterministic sequence  $\{m_k, k \in \mathbb{N}\}$ . Coup de grâce is to conclude the almost sure and weak convergence of the empirical spectral distributions from the pointwise almost sure convergence of their moments sequence; it is the province of the method of moments whose details are spelled out in [43, Appendix A3] and they can be reused word for word in the case of our empirical spectral distributions  $\mathbf{F}_n(\cdot)$ :

**Lemma 5 (Method of Moments, Theorem 2.1 of [43]).** If  $\mathbf{m}_k^{(n)} \asymp m_k$  almost surely for all  $k \in \mathbb{N}$ , then with probability one  $\mathbf{F}_n(\cdot)$  converges weakly to  $F(\cdot)$  as  $n \rightarrow \infty$ , where  $F(\cdot)$  is the unique distribution function satisfying  $\forall k \in \mathbb{N}, \int_{-\infty}^{+\infty} x^k dF(x) = m_k$ .

In fact, Theorem 2.1 in [43] provides more, and it is further true that  $\mathbf{F}_n(\cdot)$  converges weakly, almost surely, to  $F(\cdot)$  as  $n \rightarrow \infty$  (cf. [43, Appendix A1] for the definitions of the two modes of weak convergence). We have thus pieced together all the ingredients required to prove our main result, characterizing the moment sequence of the almost sure weak limit of the  $(1/\sqrt{n})$ -normalized adjacency matrix of random graph models with specified expected degree sequence.

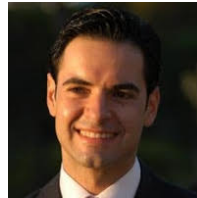
## REFERENCES

- [1] F. Chung, *Spectral Graph Theory* (CBMS Regional Conference Series in Mathematics, No. 92). American Mathematical Society, Dec. 1996.
- [2] D. M. Cvetkovic, M. Doob, and H. Sachs, *Spectra of graphs: Theory and application*. Academic press New York, 1980, vol. 413.
- [3] D. Cvetković, P. Rowlinson, and S. Simić, "An introduction to the theory of graph spectra," Cambridge-New York, 2010.
- [4] N. Biggs, *Algebraic Graph Theory*, 2nd ed. Cambridge: Cambridge University Press, 1993.
- [5] R. Merris, "Laplacian matrices of graphs: a survey," *Linear algebra and its applications*, vol. 197, pp. 143–176, 1994.
- [6] P. Van Mieghem, J. Omic, and R. Kooij, "Virus spread in networks," *Networking, IEEE/ACM Transactions on*, vol. 17, no. 1, pp. 1–14, Feb 2009.
- [7] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Phys. Rev. Lett.*, vol. 80, no. 10, pp. 2109–2112, March 1998.
- [8] L. Lovász, "Random walks on graphs: A survey," *Combinatorics, Paul erdos is eighty*, vol. 2, no. 1, pp. 1–46, 1993.
- [9] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *Automatic Control, IEEE Transactions on*, vol. 51, no. 3, pp. 401–420, 2006.
- [10] N. A. Lynch, *Distributed algorithms*. Morgan Kaufmann, 1996.
- [11] A. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999. [Online]. Available: <http://www.sciencemag.org/cgi/content/abstract/286/5439/509>
- [12] D. Watts and S. Strogatz, "Collective dynamics of small-world networks," *Nature*, no. 393, pp. 440–442, 1998.
- [13] M. Faloutsos, P. Faloutsos, and C. Faloutsos, "On power-law relationships of the internet topology," in *SIGCOMM*, 1999, pp. 251–262.
- [14] A. Clauset, C. R. Shalizi, and M. E. J. Newman, "Power-law distributions in empirical data," *SIAM review*, vol. 51, no. 4, pp. 661–703, 2009.
- [15] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *science*, vol. 286, no. 5439, pp. 509–512, 1999.
- [16] P. Erdős and A. Rényi, "On random graphs. I," *Publ. Math. Debrecen*, vol. 6, pp. 290–297, 1959.
- [17] —, "On the evolution of random graphs," *Publ. Math. Inst. Hungar. Acad. Sci.*, vol. 5, pp. 17–61, 1960.
- [18] D. J. Watts and S. H. Strogatz, "Collective dynamics of small-world networks," *nature*, vol. 393, no. 6684, pp. 440–442, 1998.
- [19] E. A. Bender and E. R. Canfield, "The asymptotic number of labeled graphs with given degree sequences," *Journal of Combinatorial Theory, Series A*, vol. 24, no. 3, pp. 296–307, 1978.
- [20] M. Molloy and B. Reed, "The size of the giant component of a random graph with a given degree sequence," *Combinatorics, probability and computing*, vol. 7, no. 03, pp. 295–305, 1998.
- [21] S. Chatterjee, P. Diaconis, A. Sly et al., "Random graphs with a given degree sequence," *The Annals of Applied Probability*, vol. 21, no. 4, pp. 1400–1435, 2011.
- [22] F. Chung and L. Lu, "Connected components in random graphs with given expected degree sequences," *Annals of combinatorics*, vol. 6, no. 2, pp. 125–145, 2002.
- [23] F. Chung, L. Lu, and V. Vu, "Eigenvalues of random power law graphs," *Annals of Combinatorics*, vol. 7, no. 1, pp. 21–33, 2003.

- [24] —, “Spectra of random graphs with given expected degrees,” *Proceedings of the National Academy of Sciences*, vol. 100, no. 11, pp. 6313–6318, 2003.
- [25] —, “The spectra of random graphs with given expected degrees,” *Internet Mathematics*, vol. 1, no. 3, pp. 257–275, 2003.
- [26] G. Anderson, A. Guionnet, and O. Zeitouni, *An Introduction to Random Matrices*, 1st ed. Cambridge: Cambridge University Press, 2009.
- [27] T. Tao, *Topics in random matrix theory*. Graduate Studies in Mathematics, American Mathematical Society, 2012.
- [28] G. Palla and G. Vattay, “Spectral transitions in networks,” *New Journal of Physics*, vol. 8, no. 12, p. 307, 2006.
- [29] J. N. Bandyopadhyay and S. Jalan, “Universality in complex networks: Random matrix analysis,” *Phys. Rev. E*, vol. 76, p. 026109, Aug 2007.
- [30] S. Jalan and J. N. Bandyopadhyay, “Random matrix analysis of complex networks,” *Phys. Rev. E*, vol. 76, p. 046107, Oct 2007.
- [31] I. J. Farkas, I. Derényi, A.-L. Barabási, and T. Vicsek, “Spectra of real-world graphs: Beyond the semicircle law,” *Physical Review E*, vol. 64, no. 2, p. 026704, 2001.
- [32] D. Kim and B. Kahng, “Spectral densities of scale-free networks,” *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 17, no. 2, p. 026115, 2007.
- [33] S. N. Dorogovtsev, A. V. Goltsev, J. F. F. Mendes, and A. N. Samukhin, “Spectra of complex networks,” *Phys. Rev. E*, vol. 68, p. 046109, Oct 2003.
- [34] K.-I. Goh, B. Kahng, and D. Kim, “Spectra and eigenvectors of scale-free networks,” *Phys. Rev. E*, vol. 64, p. 051903, Oct 2001.
- [35] W. Aiello, F. Chung, and L. Lu, “A random graph model for power law graphs,” *Experimental Mathematics*, vol. 10, no. 1, pp. 53–66, 2001.
- [36] F. Chung and M. Radcliffe, “On the spectra of general random graphs,” *the Electronic Journal of Combinatorics*, vol. 18, no. 1, p. P215, 2011.
- [37] L. Lu and X. Peng, “Spectra of edge-independent random graphs,” *The Electronic Journal of Combinatorics*, vol. 20, no. 4, p. P27, 2013.
- [38] R. I. Oliveira, “Concentration of the adjacency matrix and of the laplacian in random graphs with independent edges,” *arXiv preprint arXiv:0911.0600*, 2009.
- [39] V. M. Preciado, A. Jadbabaie, and G. Verghese, “Structural analysis of laplacian spectral properties of large-scale networks,” *Automatic Control, IEEE Transactions on*, vol. 58, no. 9, pp. 2338–2343, Sept 2013.
- [40] V. M. Preciado and A. Jadbabaie, “Moment-based spectral analysis of large-scale networks using local structural information,” *IEEE/ACM Transactions on Networking (TON)*, vol. 21, no. 2, pp. 373–382, 2013.
- [41] V. M. Preciado and G. C. Verghese, “Low-order spectral analysis of the kirchhoff matrix for a probabilistic graph with a prescribed expected degree sequence,” *Circuits and Systems I: Regular Papers, IEEE Transactions on*, vol. 56, no. 6, pp. 1231–1240, 2009.
- [42] P. Billingsley, *Probability and Measure*, 3rd ed. New York, NY: Wiley, 1995.
- [43] V. M. Preciado and M. A. Rahimian, “Spectral moments of random matrices with a rank-one pattern of variances,” *arXiv preprint arXiv:1409.5396*, 2015.
- [44] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin, “Spectral statistics of Erdős–Rényi graphs I: Local semicircle law,” *The Annals of Probability*, vol. 41, no. 3B, pp. 2279–2375, 2013.
- [45] —, “Spectral statistics of Erdős–Rényi graphs II: Eigenvalue spacing and the extreme eigenvalues,” *Communications in Mathematical Physics*, vol. 314, no. 3, pp. 587–640, 2012.
- [46] Z. Füredi and J. Komlós, “The eigenvalues of random symmetric matrices,” *Combinatorica*, vol. 1, pp. 233–241, 1981.
- [47] U. Feige and E. Ofek, “Spectral techniques applied to sparse random graphs,” *Random Structures & Algorithms*, vol. 27, no. 2, pp. 251–275, 2005.
- [48] E. Wigner, “Characteristic vectors of bordered matrices with infinite dimensions,” *Ann. of Math.*, vol. 62, pp. 548–564, 1955.
- [49] —, “On the distribution of the roots of certain symmetric matrices,” *Annals of Mathematics*, vol. 67, no. 2, pp. 325–327, 1958.
- [50] C. Bordenave, P. Caputo, and D. Chafai, “Spectrum of markov generators on sparse random graphs,” *Communications on Pure and Applied Mathematics*, vol. 67, no. 4, pp. 621–669, 2014.
- [51] C. Bordenave and P. Caputo, “Large deviations of empirical neighborhood distribution in sparse random graphs,” *Probability Theory and Related Fields*, vol. 163, no. 1–2, pp. 149–222, 2015.
- [52] C. Bordenave, M. Lelarge, and L. Massoulié, “Non-backtracking spectrum of random graphs: community detection and non-regular ramanujan graphs,” in *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*. IEEE, 2015, pp. 1347–1357.
- [53] F. Krzakala, C. Moore, E. Mossel, J. Neeman, A. Sly, L. Zdeborová, and P. Zhang, “Spectral redemption in clustering sparse networks,” *Proceedings of the National Academy of Sciences*, vol. 110, no. 52, pp. 20935–20940, 2013.
- [54] P. Hagmann, M. Kuran, X. Gigandet, P. Thiran, V. J. Wedeen, R. Meuli, and J.-P. Thiran, “Mapping human whole-brain structural networks with diffusion mri,” *PLoS ONE*, vol. 2, no. 7, p. e597, 2007.
- [55] P. Hagmann, L. Cammoun, X. Gigandet, R. Meuli, C. J. Honey, V. J. Wedeen, and O. Sporns, “Mapping the structural core of human cerebral cortex,” *PLoS Biol*, vol. 6, no. 7, p. e159, 07 2008.
- [56] S. S. Venkatesh, *The Theory of Probability: Explorations and Applications*. Cambridge University Press, 2012.
- [57] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, *A first course in order statistics*, ser. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1992, a Wiley-Interscience Publication.
- [58] M. E. Newman, “The structure and function of complex networks,” *SIAM review*, vol. 45, no. 2, pp. 167–256, 2003.
- [59] “Air traffic control network dataset - konekt, march 2016.” <http://konekt.uni-koblenz.de>.
- [60] W. Hoeffding, “Probability inequalities for sums of bounded random variables,” *Journal of the American Statistical Association*, vol. 58, no. 301, pp. 13–30, 1963.

## ACKNOWLEDGMENTS

This work was supported in part by the United States National Science Foundation under grants CNS-1302222 and IIS-1447470.



**Victor M. Preciado** received his Ph.D. degree in Electrical Engineering and Computer Science from the Massachusetts Institute of Technology in 2008. He is currently the Raj and Neera Singh Assistant Professor of Electrical and Systems Engineering at the University of Pennsylvania. He is a member of the Networked and Social Systems Engineering (NETS) program and the Warren Center for Network and Data Sciences. His research interests include network science, dynamic systems, control theory, and convex optimization with applications in socio-technical systems, technological infrastructure, and biological networks.



**M. Amin Rahimian** is a recipient of gold medal in 2004 Iran National Chemistry Olympiad. He was awarded an honorary admission to Sharif University of Technology, where he received his B.Sc. in Electrical Engineering-Control. In 2012 he received his M.A.Sc. from Concordia University in Montréal, and in 2016 he received his A.M. in Statistics from the Wharton School at the University of Pennsylvania, where he is currently a PhD student at the Department of Electrical and Systems Engineering and the GRASP Laboratory. He was a finalist in 2015 Facebook Fellowship Competition, as well as 2016 ACC Best Student Paper Competition. His research interests include network science, distributed control and decision theory, with applications to social and economic networks.